

# Gravitational confinement of photons and matter from Induced Matter theory.

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## Abstract

Using an Extended Schwarzschild-Anti de Sitter (ESAdS) metric here introduced, where the non-compact extra dimension is time-like, we study the gravitational confinement of matter and photons from a 5D vacuum. The important result here obtained is that the photon trajectories (in general, massless particles), are periodic. They are confined to a radius  $r \leq 2mG/c^2$  by the gravitational field. Massive test particles are also gravitationally confined, but their trajectories are not periodic.

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## I. INTRODUCTION AND MOTIVATION

The possibility that gravity may have a significant role at very short distances has attracted significant interest for many years[1–3] as well as the feasibility of developing a unified gauge theory of gravitational and strong forces[1] have been discussed. In the framework of brane theories with non-compact extra dimensions it is postulated that particles and fields of the standard model are confined to the brane universe, while the graviton is assumed to propagate in the bulk. Within the classical framework of this scenario, the confining of a test particle to the brane eliminates the effects of extra dimensions rendering them undetectable. In general, non-gravitational forces acting in the bulk (and orthogonal to the brane), are needed in order to keep the test particles moving on the brane, the source of these confining forces being interpreted in different manners. If the notion of confinement must appear in any reasonable theory with non-compact extra dimensions non-gravitational forces cannot be excluded a priori. Confinement due to oscillatory behavior has been proved in five-dimensional relativity with two times[4, 5]. Indeed, null paths of massless particles in 5D geodesic motion can appear in 4D as time-like paths of massive particles which undergo oscillations in the 5D dimension around the 4D hypersurface. Although the gravitational confinement has been studied many years ago in the framework of Kaluza-Klein theory[6], this issue remains unexplored in the framework of the Induced Matter theory of gravity[7], which is mathematically based on the Campbell-Magaard theorem[8–12].

In a previous work[13] we have considered a 5D extension of General Relativity such that the effective 4D gravitational dynamics has a vacuum dominated equation of state:  $\omega = -1$ . The starting 5D Ricci-flat metric  $g_{ab}$  is determined by the line element[13, 14]

$$dS^2 = \left(\frac{\psi}{\psi_0}\right)^2 \left[ c^2 f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \right] - \epsilon d\psi^2, \quad (1)$$

where  $f(r) = 1 - (2G\zeta\psi_0/(rc^2))[1 + \epsilon c^2 r^3/(2G\zeta\psi_0^3)]$  is a dimensionless function,  $\{t, r, \theta, \phi\}$  are the usual local spacetime spherical coordinates employed in general relativity and  $\psi$  is the space-like ( $\epsilon = 1$ ) extra dimension that following the approach of the Induced Matter theory, will be considered as non-compact. Furthermore,  $\psi$  and  $r$  have length units, meanwhile  $\theta$  and  $\phi$  are angular coordinates,  $t$  is a time-like coordinate and  $c$  denotes the speed of light. The effective 4D metric (1) with  $\epsilon = 1$ , is static, exterior and describes spherically symmetric matter (ordinary matter, dark matter and dark energy) on scales  $r_0 < r_{Sch} < c/H_0$  for black

holes or  $r_{Sch} < r < c/H_0$  for ordinary stars with  $r_0$  being the radius of the star. Furthermore, this metric describes both, gravity (for  $r < r_{ga}$ ) and antigravity (for  $r > r_{ga}$ ),  $r_{ga}$  being the radius for which the effective 4D gravitational acceleration becomes zero[13].

In this letter we shall study a new Ricci-flat metric where the extra coordinate is time-like ( $\epsilon = -1$ ). In this case the metric is an Extended Schwarzschild-Anti de Sitter one (ESAdS). We shall consider that  $\psi_0$  is an arbitrary constant with length units and the constant parameter  $\zeta$  has units of  $(mass)(length)^{-1}$ . The metric (1) is valid for  $f(r) > 0$ , so that its range of validity is

$$0 < r < \frac{3^{1/3}\psi_0}{3c} \frac{\left[ \left[ c \left( 9\xi G + \sqrt{3}\sqrt{c^4 + 27G^2\xi^2} \right) \right]^{2/3} - c^2 3^{1/3} \right]}{\left[ c \left( 9G\xi + \sqrt{3}\sqrt{c^4 + 27G^2\xi^2} \right) \right]^{1/3}}, \quad (2)$$

for  $\xi \neq 0$ .

## II. BASIC 5D EQUATIONS

For a massive test particles of a spacetime with spherical symmetry in a 5D bulk described by (1), the 5D Lagrangian can be written as<sup>1</sup>

$$^{(5)}L = \frac{1}{2}g_{ab}U^aU^b = \frac{1}{2}\left(\frac{\psi}{\psi_0}\right)^2 \left[ c^2 f(r) (U^t)^2 - \frac{(U^r)^2}{f(r)} - r^2 (U^\theta)^2 - r^2 \sin^2\theta (U^\phi)^2 \right] - \frac{\epsilon}{2} (U^\psi)^2. \quad (3)$$

We shall take  $\theta = \pi/2$ . Since  $t$  and  $\phi$  are cyclic coordinates, their related momentums,  $p_t$  and  $p_\phi$ , are constants of motion

$$p_t \equiv \frac{\partial ^{(5)}L}{\partial U^t} = c^2 \left(\frac{\psi}{\psi_0}\right)^2 f(r) U^t, \quad (4)$$

$$p_\phi \equiv \frac{\partial ^{(5)}L}{\partial U^\phi} = - \left(\frac{\psi}{\psi_0}\right)^2 r^2 U^\phi. \quad (5)$$

Using the constants of motion given by (4) and (5), we can express the five-velocity condition,  $g_{ab}U^aU^b = -nc^2$ , as follows:

$$\left(\frac{\psi_0}{\psi}\right)^2 \frac{p_t^2}{c^2 f(r)} - \left(\frac{\psi}{\psi_0}\right)^2 \frac{(U^r)^2}{f(r)} - \frac{p_\phi^2}{r^2} \left(\frac{\psi_0}{\psi}\right)^2 - \epsilon (U^\psi)^2 = -nc^2, \quad (6)$$

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<sup>1</sup> In this letter  $a, b$  run from 0 to 4 and Greek letters run from 0 to 3.

where  $n$  can take respectively the values  $n = 0, 1$  for photons and massive test particles. Using the expression for  $f(r)$  in Eq. (6), we obtain

$$\frac{1}{2} (U^r)^2 + \frac{\epsilon}{2} \left( \frac{\psi_0}{\psi} \right)^2 (U^\psi)^2 + V_{eff}(r) = E_n. \quad (7)$$

If we identify the energy  $E_n$  as

$$E_n = \frac{1}{2} \left( \frac{\psi_0}{\psi} \right)^4 (p_t^2 c^{-2} + \epsilon p_\phi^2 \psi_0^{-2}) + \frac{nc^2}{2} \left( \frac{\psi_0}{\psi} \right)^2, \quad (8)$$

the effective 5D potential  $V_{eff}(r)$  results to be

$$\begin{aligned} V_{eff}(r) = & - \left( \frac{\psi_0}{\psi} \right)^2 \frac{G\zeta\psi_0}{r} + \left( \frac{\psi_0}{\psi} \right)^4 \left[ \frac{p_\phi^2}{2r^2} - \frac{G\zeta\psi_0 p_\phi^2}{c^2 r^3} \right] \\ & - \frac{1}{2} \left( \frac{\psi_0}{\psi} \right)^2 \left[ (U^\psi)^2 \left( \frac{2G\zeta\psi_0}{c^2 r} + \epsilon \frac{r^2}{\psi_0^2} \right) + \epsilon \left( \frac{rc}{\psi_0} \right)^2 \right]. \end{aligned} \quad (9)$$

In order to study the effective 4D manifestation of the potential (9) for both, massive particles and photons, we shall study a static foliation  $\psi = \psi_0 = c/H_0$ , such that the dynamics evolves on an effective 4D manifold  $\Sigma_0$ . From the point of view of an relativistic observer, this implies that  $U^\psi = 0$ .

### III. EFFECTIVE 4D DYNAMICS

In this section we shall study the effective 4D dynamics on the hypersurface  $\psi = \psi_0 = c/H_0$ , which can be obtained by considering a frame  $U^\psi = 0$ . This fact implies from the point of view of the geodesic trajectories that they are described by the equation

$$\frac{dU^a}{dS} + \Gamma^a_{bc} U^b U^c = \phi^a, \quad (10)$$

where  $U^a = \frac{dX^a}{dS}$  and  $\phi^a$  is an external force. Furthermore, the velocity condition (6) must be fulfilled.

From the geodesic point of view, the equation (10) implies that

$$\frac{dU^\alpha}{dS} + \Gamma^\alpha_{ab} U^a U^b = \phi^\alpha, \quad (11)$$

$$\frac{dU^\psi}{dS} + \Gamma^\psi_{ab} U^a U^b = \phi^\psi_{(n)}, \quad (12)$$

where

$$\phi^\alpha = 0, \quad (13)$$

$$\phi^\psi_{(n)} = \frac{n}{\psi_0}. \quad (14)$$

In the eq. (13) we have supposed the non existence of an additional fifth force:  $\phi^\alpha = 0$ . The only non-zero force that we shall consider is  $\phi_{(n)}^\psi$ , but only for massive test particles. In this case it plays the role of a confining force.

### A. Photon trajectories on the brane

In order to consider the orbital equation for the photons, we take  $u(\phi) = 1/r(\phi)$  and using (4) and (5) in Eq. (6), we obtain the following equation for the orbits of photons on the brane:

$$\frac{d^2 u}{d\phi^2} + u \left[ 1 - \left( \frac{3mG}{c^2} \right) u \right] = 0, \quad (15)$$

where we have obtained the 4D hypersurface after taking the foliation  $\psi = \psi_0$ . A particular solution for this equation is  $u(\phi) = c^2 \frac{[\tan^2(\phi/2)+1]}{2mG}$ , so that

$$r(\phi) = \frac{2mG}{c^2 [\tan^2(\phi/2) + 1]}. \quad (16)$$

In the figure 1 we have plotted the radius as a function of  $\phi$ :  $r(\phi)$ . For simplicity we have taken  $\frac{2mG}{c^2}$ . Notice that the trajectories are closed so that photons remain on radius on the range  $0 < r(\phi) < \frac{2mG}{c^2}$ .

### B. Test massive particles

If we consider test massive particles, we obtain from eqs. (4) and (5) that the momentums  $\bar{p}_t$  and  $\bar{p}_\phi$  on the four dimensional hypersurface  $\Sigma_0$ , are

$$\bar{p}_t \equiv \frac{\partial {}^{(5)}L}{\partial U^t} \Big|_{\psi=\psi_0} = c^2 \left( \frac{\psi}{\psi_0} \right)^2 f(r) U^t \Big|_{\psi=\psi_0}, \quad (17)$$

$$\bar{p}_\phi \equiv \frac{\partial {}^{(5)}L}{\partial U^\phi} \Big|_{\psi=\psi_0} = - \left( \frac{\psi}{\psi_0} \right)^2 r^2 U^\phi \Big|_{\psi=\psi_0}. \quad (18)$$

Because we are dealing with a 5D static metric on which we make a static foliation  $\psi = \psi_0$ , we choose  $U^\psi = 0$ . Therefore, if we take this foliation in the potential (9), we obtain the effective 4D potential on the brane

$$\bar{V}_{eff}(r) = -\frac{mG}{r} + \bar{p}_\phi^2 \left[ \frac{1}{2r^2} - \frac{mG}{c^2 r^3} \right] + \frac{c^2}{2} \frac{r^2}{\psi_0^2}. \quad (19)$$

In the absence of angular moments, i.e. for  $\bar{p}_\phi = 0$ , one obtains only gravitational effects, so that the effective 4D gravitational force is

$$\vec{a} = -\vec{\nabla}\bar{V}_{eff}(r) = -\left[\frac{mG}{r^2} + rH_0^2\right]\hat{r} < 0, \quad (20)$$

which is related to the effective 4D gravitational potential with  $\bar{p}_\phi = 0$  and  $m = \xi c/H_0$ :  $\bar{V}_{eff}(r)$

$$\bar{V}_{eff}(r) = \frac{1}{2}r^2H_0^2 - \frac{Gm}{r}. \quad (21)$$

In other words,  $\xi$  is a parameter that can be obtained from the size of the horizon  $\psi_0 = c/H_0$  and the mass of the source  $m$ . Notice that the central acceleration is always attractive for any massive source. However its value takes a point of inflection at  $r = r_*$ , where  $\bar{V}_{eff}(r)|_{r_*} = 0$ , being

$$r_* = \left(\frac{2mG}{H_0^2}\right)^{1/3}. \quad (22)$$

For values  $r > r_*$  the gravitational potential (21) is positive and there is no confinement.

The effective 4D Einstein equations being given by

$$\bar{G}_\beta^\alpha = -8\pi G \bar{T}_\beta^\alpha, \quad (23)$$

such that the effective 4D energy-momentum tensor in a static frame [ $u^\alpha = e^\alpha_a U^a$ ]

$$\bar{u}^t = \bar{u}^\phi = \bar{u}^\theta = 0, \quad \bar{u}^r = \pm\sqrt{f(r)}c, \quad (24)$$

is

$$\bar{T}_\beta^\alpha = (P + \rho) \bar{u}^\alpha \bar{u}_\beta - \delta_\beta^\alpha \rho. \quad (25)$$

Since the effective 4D relevant components of the Einstein tensor are

$$\bar{G}_0^0 = \bar{G}_r^r = \frac{3}{\psi_0^2} \bigg|_{\psi_0=c/H_0} = \frac{3H_0^2}{c^2}, \quad (26)$$

we obtain that the effective 4D equation of state for the induced gravitational system is  $\omega = P/\rho = -1$ .

### C. Trajectories of massive test particles

When we go down from 5D to 4D the differential equation for the orbits of massive test particles is

$$\frac{d^2r}{d\phi^2} + \frac{r^4}{\bar{p}_\phi^2} \frac{d}{dr} [\bar{V}_{eff}(r)] + \frac{4r^3}{\bar{p}_\phi^2} \bar{V}_{eff}(r) - 2 \left[ \frac{c^2}{\bar{p}_\phi^2} + \frac{\bar{p}_t^2}{c^2 \bar{p}_\phi^2} - \frac{c^2}{H_0^2} \right] r^3 = 0, \quad (27)$$

where  $\bar{V}_{eff}(r)$  is given by (21). The general solution for this equation is

$$\phi - \phi_0 = \pm \int^{s=r(\phi)} \frac{6\psi_0 \bar{p}_\phi c \, ds}{\sqrt{36C_1 \bar{p}_\phi^2 c^2 \psi_0^2 - 120mG\psi_0^2 s^3 - 60c^4 s^6 + 18C\bar{p}_\phi^2 c^2 \psi_0^2 s^4}}, \quad (28)$$

where  $C = c^2 \left( \frac{1}{\bar{p}_\phi^2} - \frac{1}{H_0^2} \right) + \frac{1}{c^2} \frac{\bar{p}_t^2}{\bar{p}_\phi^2} > 0$ . One must require that  $36C_1 \bar{p}_\phi^2 c^2 \psi_0^2 - 120mG\psi_0^2 r^3 - 60c^4 r^6 + 18C\bar{p}_\phi^2 c^2 \psi_0^2 r^4 > 0$ , in order to the solution be real. For the special case where  $C_1 = 0$ , this condition implies that the radius of any trajectory for one massive test particle cannot exceed the value

$$0 < r < \frac{\left[ 10\sqrt{10000G^2m^2c^4 - 10\frac{c^4}{H_0^2} \left( 1 - \frac{\bar{p}_\phi^2}{H_0^2} \right) + \frac{\bar{p}_t^2}{H_0^2} - 1000Gmc^2} \right]^{1/3}}{10c^2} + \frac{\left[ \frac{c^4}{H_0^4} \left( 1 - \frac{\bar{p}_\phi^2}{H_0^2} \right) + \frac{\bar{p}_t^2}{H_0^2} \right]}{\left[ 10c^2\sqrt{10000G^2m^2c^4 - 10\frac{c^4}{H_0^2} \left( 1 - \frac{\bar{p}_\phi^2}{H_0^2} \right) + \frac{\bar{p}_t^2}{H_0^2} - 1000Gmc^2} \right]^{1/3}}, \quad (29)$$

where, in order to  $C > 0$ , we must require that

$$\left( 1 + \frac{\bar{p}_t^2}{c^2} \right) > \frac{\bar{p}_\phi^2}{H_0^2}. \quad (30)$$

All possible trajectories for massive test particles are given by the solutions of eq. (27) [with the conditions (29) and (30) included], and can be obtained by numerical methods. As can be noted, they are not periodic. However this issue goes beyond the scope of this letter.

#### IV. FINAL COMMENTS

We have introduced a new Ricci-flat 5D static ESAdS metric such that the extra coordinate is time-like and noncompact. This metric is very interesting because, after make a static foliation  $\psi = \psi_0 = c/H_0$  on the extra coordinate, we obtain an effective 4D hypersurface which describes the gravitational confinement of the massive test particles on a radius given by the condition (2). Furthermore, the effective 4D equation of state for the induced gravitational system is also  $\omega = P/\rho = -1$ , as in the case with  $\epsilon = 1$ . Notice that in the eq. (13) the only non-zero force that we shall consider for massive test particles is  $\phi_{(n)}^\psi$ , which plays the role of a confining force on the 4D hypersurface obtained after making the static foliation. However, for massless particles like photons this force is null. In this case massless particles exhibit periodic orbits, with radius  $r(\phi) \leq (2mG)/c^2$ . This can be seen in the

Fig. (1), where we have plotted the particular solutions (16), which describe the periodic orbit of photons for arbitrary initial conditions. Of course, this kind of solutions for photons demonstrates that they are confined by the gravitational field with a metric (1), in which  $\epsilon = -1$ .

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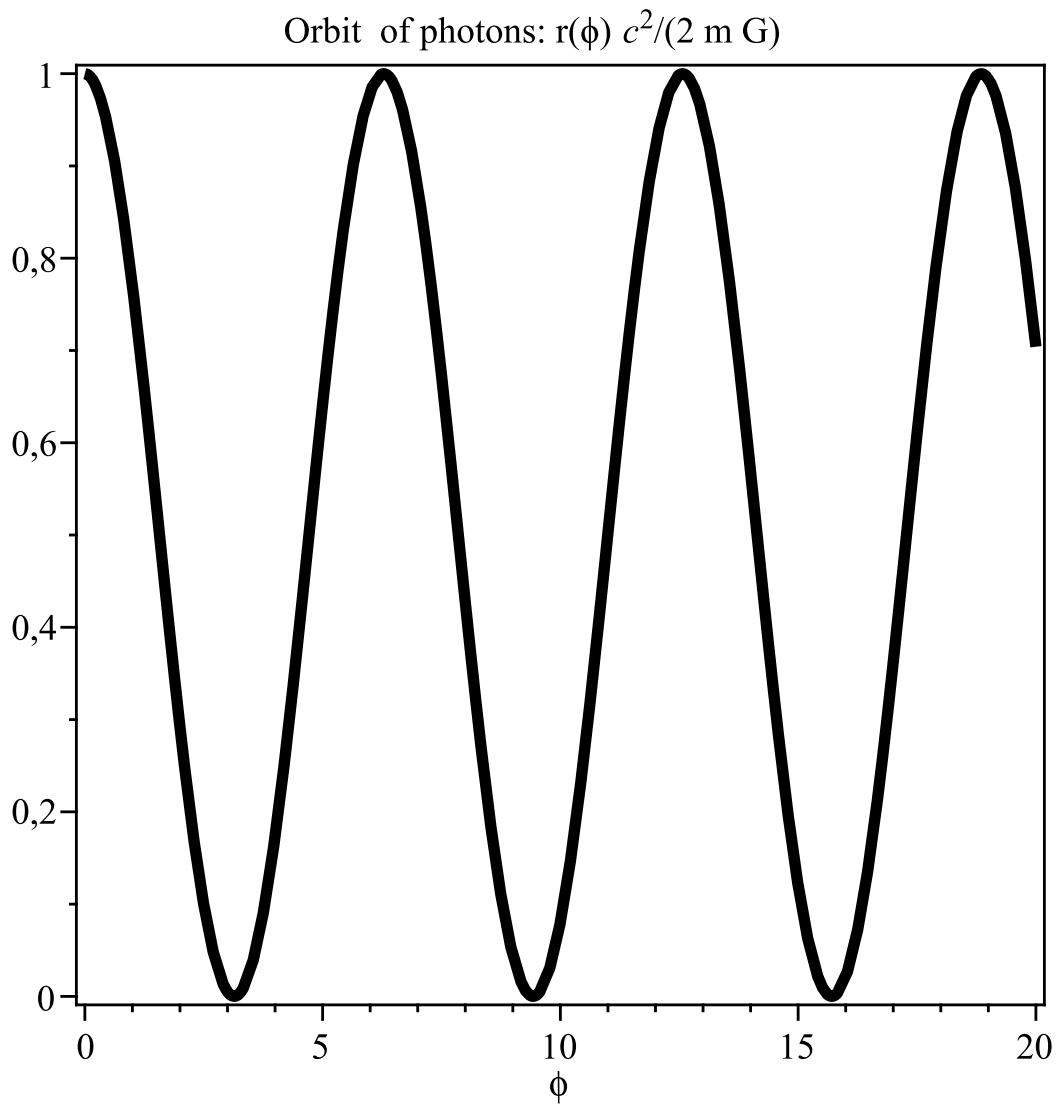


FIG. 1: Orbit of photons  $r(\phi)c^2/(2mG)$ . Notice that the periodic trajectories remain confined to  $r \leq 2mG/c^2$ .